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Renormalized expression for the turbulent energy dissipation rate

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Conditional elimination of degrees of freedom is shown to lead to an exact expression for the rate of turbulent energy dissipation in terms of a renormalized viscosity and a correction. The correction is neglected on the basis of a previous hypothesis [W. D. McComb and C. Johnston, J. Phys. A **33**, L15 (2000)] that there is a range of parameters for which a quasistochastic estimate is a good approximation to the exact conditional average. This hypothesis was tested by a perturbative calculation to second order in the local Reynolds number, and the Kolmogorov prefactor (taken as a measure of the renormalized dissipation rate) was found to reach a fixed point which was insensitive to initial values of the kinematic viscosity and to values of the spatial rescaling factor h in the range $0.4 \leqslant h \leqslant 0.8$.

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In the numerical simulation of fluid turbulence, as in other areas of computational physics, there is a practical requirement to reduce the number of degrees of freedom explicitly simulated. However, turbulence has the noteworthy requirement that any such reduction must maintain the rate at which energy is dissipated. In this Rapid Communication we put forward a method of renormalizing the energy dissipation rate.

It seems to be widely understood that any attempt to reduce the number of degrees of freedom in the theoretical description of fluid turbulence requires some form of conditional average, in which the retained modes are kept unaveraged [1]. Yet this requirement has not been recognized in most attempts to apply the dynamical renormalization group (RG) algorithm to turbulence. Normally such methods rely instead on a band-filtered unconditional average. A critical appraisal of some of the leading approaches in this area will be found in the paper by Eyink [2].

Originally, our own work, although introducing some features of the conditional average, also relied on the use of the band-filtered unconditional average [3]. Later it was recognized that a conditional average in turbulence can only be carried out as an approximation, and the two-field decomposition was introduced to separate out random and deterministic effects [4,5]. Recently, we have redefined the conditional average in the form of a limit, eliminating the need to separate into two fields [6].

The development reported here is that one of the corrections (to mode elimination) in the momentum equation, vanishes identically in the energy equation (and hence does not contribute to energy transfer), while a second correction contributes to the energy spectrum but vanishes identically in the equation for the dissipation rate.

We consider homogeneous, isotropic, incompressible, stationary turbulence, with dissipation rate ε given by

$$\varepsilon = \int_0^\infty dk 2\nu_0 k^2 E(k) \simeq \int_0^{k_0} dk 2\nu_0 k^2 E(k), \tag{1}$$

where the approximate equality defines the maximum wave number k_0. The value of k_0 is of the same order as the Kolmogorov dissipation wave number $k_d^{(0)} = (\varepsilon/\nu_0^3)^{1/4}$, where ν_0 is the kinematic viscosity.

In such turbulence the pair-correlation takes the form $\langle u_\alpha(\mathbf{k},t) u_\beta(\mathbf{k}',s)\rangle = D_{\alpha\beta}(\mathbf{k}) Q(\mathbf{k};t,s)\delta(\mathbf{k}+\mathbf{k}')$, where $D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2$, and the energy spectrum is related to the spectral density by $E(k,t) = 4\pi k^2 Q(k;t,t)$. To complete the specification of our problem, we assume that energy is being injected into some low range of wave numbers by a source term $W(k)$, which satisfies

$$\int_0^\kappa dk W(k) = \varepsilon, \tag{2}$$

for some $\kappa \ll k_d^{(0)}$. This ensures stationarity.

Next we introduce a version of the RG which leads to a renormalized dissipation rate equation. The Navier-Stokes equation (NSE) may be written in dimensionless form as

$$\{\partial_{\hat{t}} + \hat{\nu}_0(\hat{k})\hat{k}^2\}\hat{u}_\alpha(\hat{\mathbf{k}},\hat{t}) = R_0(k_0) M_{\alpha\beta\gamma}(\hat{\mathbf{k}}) \times \int d^3\hat{j}\, \hat{u}_\beta(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t}), \tag{3}$$

on $0 < \hat{k} < \hat{k}_0 = 1$, where $\hat{k} = k/k_0$, $\hat{t} = t/\tau(k_0)$, $\hat{u}_\alpha(\hat{\mathbf{k}},t) = u_\alpha(\mathbf{k},t)/V(k_0)$, $\tau(k_0)$ is, as yet, an undefined timescale, $V(k_0)$ is the rms value of a velocity mode with $|\mathbf{k}| = k_0$, defined for any k by

$$V^2(k) = (1/k^3)\langle u_\alpha(\mathbf{k},t) u_\alpha(-\mathbf{k},t)\rangle, \tag{4}$$

$R_0(k_0) = \tau(k_0)V(k_0)k_0^4$ is the local Reynolds number (see Batchelor [7], p. 107) and $M_{\alpha\beta\gamma}(\mathbf{k}) = (2i)^{-1}[k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})]$. It should also be noted that the local Reynolds number is indeed nondimensional, since $u_\alpha(\mathbf{k},t)$ has dimensions L^4T^{-1}. The dynamical RG algorithm can now be stated as follows:

(i) (Rescale all wave vectors on $\hat{k}_1$ $(= h\hat{k}_0)$, where $0 < h < 1$, for example $k' = \hat{k}/\hat{k}_1$, such that $0 < k' < h^{-1}$, and then average out the effects of the high wave number modes to obtain a dynamical equation for the modes on the interval

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$0<k'<k_1'$ $(\equiv 1)$. (ii) Use this low-wave number NSE to obtain the low-wave-number energy balance equation. (iii) Integrate the energy balance equation with respect to k' up to the value $k'=k_1'(\equiv 1)$ to derive an equation for the dissipation rate. (iv) Repeat these steps until the dissipation rate reaches a fixed point, at a new maximum wavenumber $k_N'=h^N k_0'$. The first step is to write the velocity fields as

$$\hat{u}_\alpha^{\pm}(\hat{\boldsymbol{k}},\hat{t})=V(k_1)\psi_\alpha^{\pm}(\boldsymbol{k}',t'), \tag{5}$$

where $\psi_\alpha^-(\boldsymbol{k}',t')$ is defined on $0<k'<1$ and $\psi_\alpha^+(\boldsymbol{k}',t')$ is defined on $1<k'<h^{-1}$. The combined (low-k and high-k) equation of motion then takes the form

$$\{\partial_{t'}+\nu_0'(k')k'^2\}\psi_\alpha^{\pm}(\boldsymbol{k}',t')$$

$$=R_1(k_1)M_{\alpha\beta\gamma}^{\pm}(\boldsymbol{k}')\int d^3j'\,\psi_\beta(\boldsymbol{j}',\boldsymbol{t}')\psi_\gamma(\boldsymbol{k}'-\boldsymbol{j}',\boldsymbol{t}'), \tag{6}$$

where $R_1(k_1)=\tau(k_1)V(k_1)k_1^4$. Next we average out the effect of the high-k modes, while leaving the low-k modes unaffected. In general this will require the conditional projection of some functional $\mathcal{F}[\psi_\alpha]$ on the ψ_α^-, which we denote by a subscript ''c,'' viz. $\langle\cdot\rangle_c$. This should not be confused with the usual ensemble average, as denoted by $\langle\cdot\rangle$. An important property of the conditional average is that the constraint is lifted by a further *unconditional* average [8].

Taking the low-k equation, as given by Eq. (6), we conditionally average both sides, and decompose the right-hand side according to Eq. (5), to obtain

$$\{\partial_{t'}+\nu_0'(k')k'^2\}\psi_\alpha^-(\boldsymbol{k}')$$

$$=R_1(k_1)M_{\alpha\beta\gamma}^-(\boldsymbol{k}')\int d^3j'\{\langle\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}')\rangle_c$$

$$+2\langle\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\rangle_c+\langle\psi_\beta^+(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\rangle_c\}. \tag{7}$$

We shall see presently that only the last term on the right-hand side (RHS) of Eq. (7) contributes to the energy dissipation rate. In [6] we gave a method for the approximate calculation of this conditional average. We shall refer to this approximation as the *quasistochastic estimate* (QSE) of the conditional average and denote it by $\langle\cdot\rangle_{QSE}$. Then, rearranging Eq. (7) and adding and subtracting quantities to leave it unaffected, we may write the low-wave-number equation as

$$(\partial_{t'}+\nu_0'(k')k'^2)\psi_\alpha^-(\boldsymbol{k}')$$

$$-R_1(k_1)M_{\alpha\beta\gamma}^-(\boldsymbol{k}')\int d^3j'\langle\psi_\beta^+(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\rangle_{QSE}$$

$$=R_1(k_1)M_{\alpha\beta\gamma}^-(\boldsymbol{k}')\int d^3j'\,\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}')$$

$$+S_\alpha^-(\boldsymbol{k}'|k_1'), \tag{8}$$

where

$$S_\alpha^-(\boldsymbol{k}'|k_1')$$

$$=R_1(k_1)M_{\alpha\beta\gamma}^-(\boldsymbol{k}')$$

$$\times\int d^3j'\{\underbrace{\langle\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}')\rangle_c-\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}')}_{S_1}$$

$$+2\underbrace{\langle\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\rangle_c}_{S_2}$$

$$+\underbrace{\langle\psi_\beta^+(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\rangle_c-\langle\psi_\beta^+(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\rangle_{QSE}}_{S_3}\}. \tag{9}$$

We now wish to obtain the QSE, $\langle\psi_\beta^+(\boldsymbol{j}',t')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}',t')\rangle_{QSE}$, which appears on the LHS of Eq. (8). Following the procedure in [6], we form an equation of motion for this quantity from Eq. (6). This is solved perturbatively as a power series in $R(k_1)$ and the band-filtered moments of the ψ^+. As a result, the low-k equation, after eliminating the first band of modes, takes the form

$$\{\partial_{t'}+\nu_0'(k')k'^2\}\psi_\alpha^-(\boldsymbol{k}')-\int ds'A_{\alpha\beta}(\boldsymbol{k}',t'-s')\psi_\beta^-(\boldsymbol{k}',s')$$

$$=R_1(k_1)M_{\alpha\beta\gamma}^-(\boldsymbol{k}')\int d^3j'\langle\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}')\rangle_c$$

$$+S_\alpha^-(\boldsymbol{k}'|k_1'), \tag{10}$$

where

$$A_{\alpha\beta}(\boldsymbol{k}',t'-s')=D_{\alpha\beta}(\boldsymbol{k}')R_1^2(k_1)[A^{(0)}(k',t'-s')$$

$$+A^{(1)}(k',t'-s')R_1(k_1)+A^{(2)}$$

$$\times(k',t'-s')R_1^2(k_1)+\cdots]. \tag{11}$$

The coefficients $A^{(0)},A^{(1)},A^{(2)},\ldots$ depend on the moments of ψ^+ of order 2,3,4,... respectively. It should also be noted that the even-order coefficients are real and the odd-order are imaginary, since the expansion is effectively in powers of $M_{\alpha\beta\gamma}(\boldsymbol{k})$, which is imaginary.

We now form the energy balance equation for the explicit scales $k\leqslant k_1$, by multiplying each side of Eq. (10) through by $\psi_\alpha^-(-\boldsymbol{k}',t')$ and averaging unconditionally. We then multiply through by appropriate factors to restore the original unscaled variables, in the process introducing the energy spectrum $E(k)$. We also add $W(k)$, as specified in Eq. (2), with the result

$$(\partial_t+2\nu_0(k)k^2)E(k)+2\int dsA(k,t-s)E(k,s)$$

$$=W(k)+T(k)+8\pi k^2V^2(k_1)\langle S_\alpha^-(\boldsymbol{k}|k_1)\psi_\alpha^-(-\boldsymbol{k})\rangle, \tag{12}$$

where $A(k)=trA_{\alpha\beta}(\boldsymbol{k})$ and $T(k,t)=\int dj\tilde{T}(k,j,|\boldsymbol{k}-\boldsymbol{j}|;t)$ is the usual transfer spectrum, with wave numbers in the interval $0\leqslant k,j,\ |\boldsymbol{k}-\boldsymbol{j}|\leqslant k_1$.

Lastly, we may form an equation for the rate at which energy is transferred through the modes of the system. Integrating Eq. (12) with respect to k, we obtain

$$2\int_0^{k_1} dk[\nu_0 k^2 + A(k)]E(k) = \varepsilon + 8\pi \int_0^{k_1} dk k^2 V^2(k_1)\langle S_\alpha^-(\boldsymbol{k}|k_1)\psi_\alpha^-(-\boldsymbol{k})\rangle. \quad (13)$$

Note that the integral over the transfer term vanishes identically due to the antisymmetry of $\tilde{T}(k,j,|\boldsymbol{k}-\boldsymbol{j}|;t)$ under interchange of $\boldsymbol{k}$ and $\boldsymbol{j}$ (see [7], p. 85).

At this stage, all three renormalized conservation equations (for momentum, energy, and dissipation rate) are exact. Now consider the effect of the term S_α^-, divided into three parts, as shown in Eq. (9), and begin with the energy equation. The conditional average behaves as a stochastic variable under a further *unconditional* average [8]. Thus for the first term we have (schematically)

$$\langle S_1\psi_\alpha^-\rangle \sim \int d^3j'\{\langle\langle\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}')\rangle_c\psi_\alpha^-(-\boldsymbol{k}')\rangle - \langle\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}')\psi_\alpha^-(-\boldsymbol{k}')\rangle\} = 0, \quad (14)$$

and so the contribution from S_1 vanishes identically in the energy equation.

Now we turn to the dissipation equation: evidently, we need only consider S_2 and S_3. The first of these gives

$$\int d^3k'\langle S_2\psi_\alpha^-\rangle \sim \int d^3k' \int d^3j'\langle\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\psi_\alpha^-(-\boldsymbol{k}')\rangle = 0, \quad (15)$$

by antisymmetry under interchange of $\boldsymbol{k}'$ and $\boldsymbol{j}'$. It should be noted that this property holds only because both wave numbers are on the same interval. This is not the case regarding the contribution from S_3, which is

$$\int d^3k'\langle S_3\psi_\alpha^-\rangle \sim \int d^3k' \int d^3j'\{\langle\psi_\beta^+(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\psi_\alpha^-(-\boldsymbol{k}')\rangle - \langle\psi_\beta^+(\boldsymbol{j}')\psi_\gamma^+(\boldsymbol{k}'-\boldsymbol{j}')\psi_\alpha^-(-\boldsymbol{k}')\rangle_{QSE}\}. \quad (16)$$

However, note that the two terms will cancel under any circumstances in which the QSE is a good model for the exact conditional average.

Now, in order to perform an RG-style iteration, we truncate the expansion for $A_{\alpha\beta}$, as given by Eq. (11), at lowest order and rename

$$R_1^2(k_1)A^{(0)}(k',0) = \delta\nu_0(k')k'^2. \quad (17)$$

Note that $A^{(1)}$ is imaginary and therefore cannot contribute to the dissipation rate. This means that we effectively neglect terms of order $R_1^4(k_1)$ and higher. To this level of approximation, Eq. (10) can be written as

$$(\partial_{t'} + \nu_1'(k')k'^2)\psi_\alpha^-(\boldsymbol{k}') = S_\alpha^-(\boldsymbol{k}'|k_1') + R_1(k_1)M_{\alpha\beta\gamma}^-(\boldsymbol{k}') \times \int d^3j'\psi_\beta^-(\boldsymbol{j}')\psi_\gamma^-(\boldsymbol{k}'-\boldsymbol{j}'), \quad (18)$$

for $0<k'<1$. The renormalized viscosity is given by

$$\nu_1'(k') = \nu_0'(k') + \delta\nu_0'(k'), \quad (19)$$

and the equation for the increment is

$$\delta\nu_0'(k') = R_1^2(k_1) \lim_{\ell'\to h^{-1}} \Delta_0(k')\left[\frac{(k'^2/2) + j'^2 - k'j'\mu}{j'^2 - k'j'\mu}\right] + \mathcal{O}[R_1^4(k_1)], \quad (20)$$

where μ is the cosine of the angle between $\boldsymbol{k}'$ and $\boldsymbol{j}'$ and $\Delta_0(k')$ is the two-field form of the increment [5], given by

$$\Delta_0(k') = \frac{1}{k'^2}\int d^3j' \frac{L(\boldsymbol{k}',\boldsymbol{j}')\hat{Q}^+(\ell')}{\nu_0'(j')j'^2 + \nu_0'(\ell')\ell'^2}. \quad (21)$$

Here $\ell' = |\boldsymbol{k}'-\boldsymbol{j}'|$, $1<k'$, $\ell'<h^{-1}$ and $L(\boldsymbol{k}',\boldsymbol{j}') = -2M_{\delta\beta\gamma}^-(\boldsymbol{k}')M_{\beta\delta\epsilon}^+(\boldsymbol{j}')D_{\epsilon\gamma}^+(\boldsymbol{k}'-\boldsymbol{j}')$. Equation (20) for the increment to the viscosity involves $\lim|\boldsymbol{k}'-\boldsymbol{j}'|\to h^{-1}$ (see [6]), and if this limit is evaluated by taking the $\hat{Q}^+(|\boldsymbol{k}'-\boldsymbol{j}'|)$ as an expansion in Taylor series about $k'=1$, then we make contact with the two-field version of McComb and Watt [5], with spectral density given by

$$\hat{Q}(l') = \frac{1}{k_1 V^2(k_1)}\left\{h^{11/3} - \frac{11}{3}h^{14/3}(l'-h^{-1})\right\}. \quad (22)$$

It should be noted, however, that the factor in square brackets in Eq. (20) is new. This arises because we were able to improve on the Markovian approximation used in the earlier calculations of McComb *et al.* [3,5]. Details of this analysis will be given elsewhere. The equations for any iteration labeled n can be found by induction, and numerical calculation shows that the renormalized viscosity approaches a fixed point for some $n=N$, where in practice $N=5$ or 6, for most values of the spatial rescaling factor. At the fixed point, Eq. (13) may be rearranged to give

$$\varepsilon = 2\int_0^{k_N} dk\,\nu_N(k)k^2E(k) - 8\pi\int_{k_{N-1}}^{k_N} dk k^2V^2(k_N) \times\langle S_3\psi_\alpha^-(-\boldsymbol{k})\rangle, \quad (23)$$

where $\langle S_3\psi_\alpha^-(-\boldsymbol{k})\rangle$ is shown schematically in Eq. (16).

Let us now consider how to assess this work. We begin by noting that the renormalized ''viscous term'' in Eq. (18) is not an observable, even though it may be calculated to any

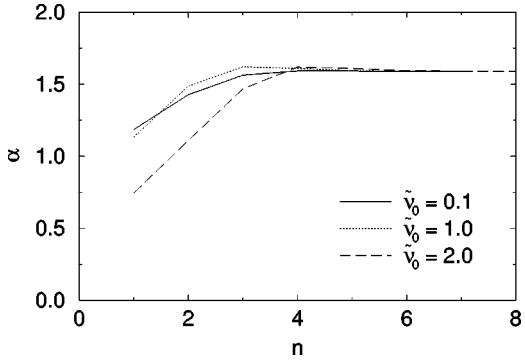

FIG. 1. Kolmogorov prefactor α reaching a fixed point for a variety of starting viscosities $\tilde{\nu}_0$. (For the case where the spatial rescaling factor $h=0.60$ or the bandwidth $\eta=0.40$.)

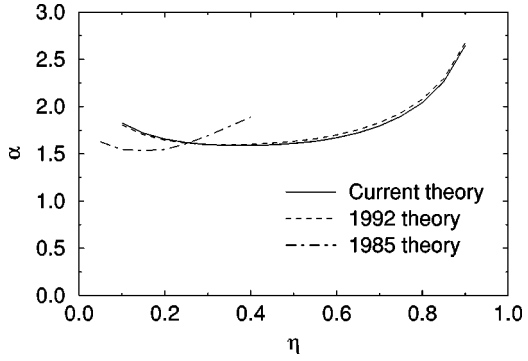

FIG. 2. Variation of the Kolmogorov prefactor α with bandwidth η or spatial rescaling factor h.

order using Eq. (11). This is because S_α^- varies from realization to realization of the explicit scales. In contrast, the renormalized viscosity in Eq. (12) would be [if we took the step given in Eq. (17)] an observable, as all terms in the equation have been averaged. However, the contribution from S_2 is likely to prove important for energy transfer and, as this has been omitted from our calculation, we shall leave discussion of this to a fuller account, and concentrate on Eq. (23) for the dissipation rate, as we know that S_2 does not contribute to this equation.

In order to calculate the renormalized dissipation rate, we assume a power-law form for the spectrum $E(k)=\alpha\varepsilon^r k^s$. Then, the requirement that the renormalized viscosity (19) and its increment (20) scale in the same way, along with the conservation requirement of Eq. (23), yield $r=2/3$ and $s=-5/3$, along with an expression for the Kolmogorov prefactor α [see Eq. (92) in Ref. [5]].

The theoretical prediction of α can be taken as a measure of the predicted dissipation rate, and in Fig. 1 we show the result of such calculations, with α iterating to a fixed point for several different starting conditions at one value of the spatial rescaling factor. The fixed point corresponds to the upper end of the inertial range and the value of α at the fixed point agrees well with the result obtained from numerical simulations. Figure 2 shows the fixed-point value of α for a range of spatial rescaling factors. It is of interest to note that the dashed-dotted curve to the left depicts an earlier version of the theory, in which the conditional average was approximated by a band-filtered average [3]. The dashed line shows the result of working out the limiting form of the viscosity (the stochastic part) with scale separation [5], while invoking a Markovian approximation. The solid line shows the effect of including the square brackets in Eq. (20) and gives rise to a prediction of $\alpha=1.62\pm0.02$ for $0.2<\eta<0.6$, where the bandwidth $\eta=1-h$. Incidentally, it is perhaps worth remarking that the limit $\eta\to0$ (which one would expect in the microscopic case) does not exist for macroscopic turbulence. This is a consequence of an exact symmetry of the NSE: local energy transfer vanishes when the wave vector triad takes the form of an equilateral triangle.

Last, there is the question, How good is our perturbation calculation? The expansion, which we truncate, is in powers of $\lambda=R_n^2(k)$, with integrals over moments of the ψ^+ (where $|\psi^+|_{\rm rms}\leqslant1$). With the maximum value $\lambda=0.16$, this is a small parameter, but possibly not small enough for the truncation to qualify as a rational approximation [9]. Accordingly, we may have to rely on the properties of the moment expansion. Certainly, the next step is to work out the magnitude of the fourth-order term. This is the subject of current work.

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